Quiz 1: Numerical Computations

This quiz requires performing calculations exactly, with three-digit chopping, and with three-digit rounding, followed by computing the relative errors.

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a. $\frac{4}{5} \times \frac{1}{3}$

- **(i) Exact:** $\frac{4}{5} \times \frac{1}{3} = \frac{4}{15} = 0.2\overline{6}$
- **(ii) Three-Digit Chopping:**
  - $\frac{4}{5} = 0.8$ and $\frac{1}{3} \approx 0.333$
  - $0.8 \times 0.333 = 0.2664$, which chops to 0.266.
- **(iii) Three-Digit Rounding:**
  - $\frac{4}{5} = 0.8$ and $\frac{1}{3} \approx 0.333$
  - $0.8 \times 0.333 = 0.2664$, which rounds to 0.266.
- **(iv) Relative Errors:**
  - The exact value is $p = \frac{4}{15}$. Both approximations are $p^* = 0.266$.
  - Relative Error = $\frac{|p-p^*|}{|p|} = \frac{|\frac{4}{15}-0.266|}{\frac{4}{15}} = \frac{|0.2\overline{6}-0.266|}{0.2\overline{6}} = \frac{0.000\overline{6}}{0.2\overline{6}} = \frac{1}{400} = \mathbf{0.0025}$

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b. $(\frac{1}{3} + \frac{3}{11}) - \frac{3}{20}$

- **(i) Exact:** $(\frac{1}{3} + \frac{3}{11}) - \frac{3}{20} = \frac{20}{33} - \frac{3}{20} = \frac{400-99}{660} = \frac{301}{660} \approx 0.45\overline{60}$
- **(ii) Three-Digit Chopping:**
  - $\frac{1}{3} \approx 0.333$, $\frac{3}{11} \approx 0.272$, $\frac{3}{20} = 0.150$
  - $(0.333 + 0.272) - 0.150 = 0.605 - 0.150 = \mathbf{0.455}$
- **(iii) Three-Digit Rounding:**
  - $\frac{1}{3} \approx 0.333$, $\frac{3}{11} \approx 0.273$, $\frac{3}{20} = 0.150$
  - $(0.333 + 0.273) - 0.150 = 0.606 - 0.150 = \mathbf{0.456}$
- **(iv) Relative Errors:**
  - Exact value $p = \frac{301}{660}$.
  - Chopping Error: $\frac{|\frac{301}{660}-0.455|}{\frac{301}{660}} \approx \frac{0.0010606}{0.4560606} \approx \mathbf{0.002325}$
  - Rounding Error: $\frac{|\frac{301}{660}-0.456|}{\frac{301}{660}} \approx \frac{0.0000606}{0.4560606} \approx \mathbf{0.000133}$

Quiz 2: Root of a Polynomial

To find a solution for $x^3 - x - 1 = 0$ on the interval $[1, 2]$ with an error bound of 10^{-2} using the fixed-point iteration method and an initial guess of $p_0 = 1$. [cite: 4]

1. Rearrange the equation into the form $x = g(x)$. A suitable choice that satisfies the convergence criteria on $[1, 2]$ is $x = (x+1)^{1/3}$. Let $g(x) = (x+1)^{1/3}$.
2. Verify convergence: The derivative is $g'(x) = \frac{1}{3(x+1)^{2/3}}$. On $[1, 2]$, $|g'(x)| \leq |g'(1)| \approx 0.21 < 1$, which ensures convergence.
3. Iterate starting with $p_0 = 1$:
   - $p_1 = g(p_0) = (1+1)^{1/3} = 2^{1/3} \approx 1.2599$

- $p_2 = g(p_1) = (1.2599 + 1)^{1/3} \approx 1.3123$
- $p_3 = g(p_2) = (1.3123 + 1)^{1/3} \approx 1.3224$

4. **Check the error**: We can use the error estimate $|p - p_n| \le \frac{k}{1-k}|p_n - p_{n-1}|$.
   - For $n = 3$, the error is bounded by approximately $\frac{0.21}{1-0.21}|1.3224 - 1.3123| \approx 0.266 \times 0.0101 \approx 0.0027$.
   - Since $0.0027 < 10^{-2}$, the approximation is within the required error bound.

A solution with the desired accuracy is 1.3224.

Quiz 3: Solve Linear Systems

This quiz asks to solve two linear systems using Gaussian Elimination and to determine if row interchanges are needed.

(a)

$x_1 + x_2 + x_4 = 2$
$2x_1 + x_2 - x_3 + x_4 = 1$
$4x_1 - x_2 - 2x_3 + 2x_4 = 0$
$3x_1 - x_2 - x_3 + 2x_4 = -3$

Performing Gaussian elimination on the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ 4 & -1 & -2 & 2 & 0 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & -5 & -2 & -2 & -8 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

Subtracting the third row from the fourth row yields:

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 & 7 \\ 0 & 0 & 0 & 0 & -4 \end{array}\right)$$

The last row implies $0 = -4$, which is a contradiction.

- **Solution: No solution** exists for this system.
- **Row Interchanges:** Not necessary.

(b)

$x_1 + x_2 + x_4 = 2$
$2x_1 + x_2 - x_3 + x_4 = 1$
$-x_1 + 2x_2 + 3x_3 - 4x_4 = 4$
$3x_1 - x_2 - x_3 + 2x_4 = -3$

Performing Gaussian elimination:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 & 1 \\ -1 & 2 & 3 & -4 & 4 \\ 3 & -1 & -1 & 2 & -3 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & -1 & -3 \\ 0 & 3 & 3 & -3 & 6 \\ 0 & -4 & -1 & -1 & -9 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & -6 & -3 \\ 0 & 0 & 3 & 3 & 3 \end{array}\right)$$

To proceed, a row interchange is necessary. Swapping rows 3 and 4:

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & -6 & -3 \end{array}\right)$$

Using back substitution:

- $-6x_4 = -3 \implies x_4 = \frac{1}{2}$
- $3x_3 + 3(\frac{1}{2}) = 3 \implies 3x_3 = \frac{3}{2} \implies x_3 = \frac{1}{2}$
- $x_2 + \frac{1}{2} + \frac{1}{2} = 3 \implies x_2 = 2$
- $x_1 + 2 + \frac{1}{2} = 2 \implies x_1 = -\frac{1}{2}$
- **Solution:** $x_1 = -1/2$, $x_2 = 2$, $x_3 = 1/2$, $x_4 = 1/2$.
- **Row Interchanges: Necessary**.

Here are the solutions to quizzes 4, 5, and 6 from the provided document.

Quiz 4: Matrix Factorization

The goal is to find a factorization of the form $A = P^T LU$ for the given matrix A. This involves using Gaussian elimination with partial pivoting.

The given matrix is:

$$A = \begin{pmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{pmatrix}$$

During the first step of elimination, the pivot element in the second row is zero, which requires a row interchange. We swap the second and fourth rows to get a non-zero pivot. This row interchange is captured by the permutation matrix P.

The resulting factorization is:

- **P^T (transpose of the permutation matrix):**
  $$P^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$
  (This corresponds to swapping rows 2 and 4).
- **L (lower triangular matrix):**
  $$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$
- **U (upper triangular matrix):**
  $$U = \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & 5 & -3 & -1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Quiz 5: SOR

This problem asks for the first iteration of the Successive Over-Relaxation (SOR) method with a relaxation parameter $\omega = 1.1$ for the given linear system, starting with $x^{(0)} = (0, 0, 0)^T$.

The linear system is:
$3x_1 - x_2 + x_3 = 1$
$3x_1 + 6x_2 + 2x_3 = 0$
$3x_1 + 3x_2 + 7x_3 = 4$

The SOR formula for a component x_i in iteration (k) is:
$x_i^{(k)} = (1-\omega)x_i^{(k-1)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij}x_j^{(k)} - \sum_{j>i} a_{ij}x_j^{(k-1)}\right)$

Applying this for the first iteration ($k=1$):

1. **Calculate $x_1^{(1)}$:**
   $x_1^{(1)} = (1-1.1)x_1^{(0)} + \frac{1.1}{3}(1-(-1)x_2^{(0)} - (1)x_3^{(0)})$
   $x_1^{(1)} = 0 + \frac{1.1}{3}(1-0-0) = \frac{1.1}{3} \approx \mathbf{0.367}$
2. **Calculate $x_2^{(1)}$:**
   $x_2^{(1)} = (1-1.1)x_2^{(0)} + \frac{1.1}{6}(0-(3)x_1^{(1)} - (2)x_3^{(0)})$
   $x_2^{(1)} = 0 + \frac{1.1}{6}(0-3(\frac{1.1}{3})-0) = \frac{1.1}{6}(-1.1) = -\frac{1.21}{6} \approx \mathbf{-0.202}$
3. **Calculate $x_3^{(1)}$:**
   $x_3^{(1)} = (1-1.1)x_3^{(0)} + \frac{1.1}{7}(4-(3)x_1^{(1)} - (3)x_2^{(1)})$
   $x_3^{(1)} = 0 + \frac{1.1}{7}(4-3(\frac{1.1}{3})-3(-\frac{1.21}{6})) = \frac{1.1}{7}(4-1.1+0.605) = \frac{1.1\times 3.505}{7} \approx \mathbf{0.551}$

The first iteration vector is $x^{(1)} \approx (0.367, -0.202, 0.551)^T$.

Quiz 6: Condition Number

The task is to compute the condition numbers of two matrices relative to the infinity norm, $||\cdot||_\infty$. The condition number is defined as $K(A) = ||A||_\infty ||A^{-1}||_\infty$.

(a) For the matrix $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$:

1. **Find $||A||_\infty$**: This is the maximum absolute row sum.
   - Row 1 sum: $|\frac{1}{2}| + |\frac{1}{3}| = \frac{3+2}{6} = \frac{5}{6}$
   - Row 2 sum: $|\frac{1}{3}| + |\frac{1}{4}| = \frac{4+3}{12} = \frac{7}{12}$
   - $||A||_\infty = \max(\frac{5}{6}, \frac{7}{12}) = \frac{5}{6}$
2. **Find A^{-1}**:
   - $\det(A) = (\frac{1}{2})(\frac{1}{4}) - (\frac{1}{3})(\frac{1}{3}) = \frac{1}{8} - \frac{1}{9} = \frac{1}{72}$
   - $A^{-1} = 72\begin{pmatrix} \frac{1}{4} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 18 & -24 \\ -24 & 36 \end{pmatrix}$
3. **Find $||A^{-1}||_\infty$**:
   - Row 1 sum: $|18| + |-24| = 42$
   - Row 2 sum: $|-24| + |36| = 60$
   - $||A^{-1}||_\infty = 60$
4. **Calculate the condition number**:
   - $K(A) = ||A||_\infty ||A^{-1}||_\infty = (\frac{5}{6}) \times 60 = \mathbf{50}$

(b) For the matrix $A = \begin{pmatrix} 1 & 2 \\ 1.00001 & 2 \end{pmatrix}$:

1. **Find $||A||_\infty$**:
   - Row 1 sum: $|1| + |2| = 3$
   - Row 2 sum: $|1.00001| + |2| = 3.00001$
   - $||A||_\infty = 3.00001$
2. **Find A^{-1}**:
   - $\det(A) = (1)(2) - (2)(1.00001) = 2 - 2.00002 = -0.00002$

- $A^{-1} = \frac{1}{-0.00002}\begin{pmatrix} 2 & -2 \\ -1.00001 & 1 \end{pmatrix} = \begin{pmatrix} -100000 & 100000 \\ 50000.5 & -50000 \end{pmatrix}$

3. **Find $||A^{-1}||_\infty$:**
   - Row 1 sum: $|-100000| + |100000| = 200000$
   - Row 2 sum: $|50000.5| + |-50000| = 100000.5$
   - $||A^{-1}||_\infty = 200000$
4. **Calculate the condition number:**
   - $K(A) = (3.00001) \times (200000) =$ **600,002**
     (This high condition number indicates that the matrix is ill-conditioned).

Quiz 7: Interpolation

To approximate $f(0.05)$ using the Newton forward divided-difference formula, we first construct a divided-difference table from the given data.

Data:

- $x_0 = 0.0, f(x_0) = 1.00000$
- $x_1 = 0.2, f(x_1) = 1.22140$
- $x_2 = 0.4, f(x_2) = 1.49182$
- $x_3 = 0.6, f(x_3) = 1.82212$
- $x_4 = 0.8, f(x_4) = 2.22554$

Divided-Difference Table:

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, \ldots, x_{i+2}]$	$f[x_i, \ldots, x_{i+3}]$	$f[x_i, \ldots, x_{i+4}]$
0.0	1.00000				
		1.10700			
0.2	1.22140		0.61275		
		1.35210		0.22625	
0.4	1.49182		0.74850		0.06198
		1.65150		0.27583	
0.6	1.82212		0.91400		
		2.01710			
0.8	2.22554				

The Newton forward divided-difference formula is:
$P_4(x) = f[x_0] + \sum_{k=1}^{4} f[x_0, \ldots, x_k] \prod_{i=0}^{k-1}(x - x_i)$

Using the coefficients from the top diagonal of the table (in bold):
$P_4(x) = 1.00000 + 1.10700(x - 0.0) + 0.61275(x - 0.0)(x - 0.2) + 0.22625(x - 0.0)(x - 0.2)(x - 0.4) + 0.06198$

Now, we evaluate this polynomial at $x = 0.05$:

- $P_4(0.05) = 1.00000 + 1.10700(0.05) + 0.61275(0.05)(-0.15) + 0.22625(0.05)(-0.15)(-0.35) + 0.06198(0.05)($
- $P_4(0.05) = 1.00000 + 0.05535 - 0.004595625 + 0.0005938125 - 0.000089469375$
- $P_4(0.05) \approx 1.05126$

Thus, the approximation of $f(0.05)$ is 1.05126.

Quiz 8: Least Squares Approximation

The goal is to find the least squares approximation for the spring constant k from Hooke's Law, $F(l) = k(l - E)$. We are given $E = 5.3$ and a set of measurements for force $F(l)$ and length l.

The model is $F = kx$, where $x = l - E$. We want to find the value of k that minimizes the sum of squared errors, $S = \sum_{i=1}^{n} [F_i - kx_i]^2$.

To minimize S, we set its derivative with respect to k to zero:
$\frac{dS}{dk} = \sum_{i=1}^{n} -2x_i(F_i - kx_i) = 0$
$k \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} F_i x_i$
$k = \frac{\sum_{i=1}^{n} F_i x_i}{\sum_{i=1}^{n} x_i^2}$

First, we calculate the values for $x_i = l_i - E$ using $E = 5.3$:

F_i	l_i	$x_i = l_i - 5.3$
2	7.0	1.7
4	9.4	4.1
6	12.3	7.0

Next, we calculate the sums required for the formula for k:

- $\sum F_i x_i = (2 \times 1.7) + (4 \times 4.1) + (6 \times 7.0) = 3.4 + 16.4 + 42.0 = 61.8$
- $\sum x_i^2 = (1.7)^2 + (4.1)^2 + (7.0)^2 = 2.89 + 16.81 + 49.0 = 68.7$

Now, we can compute k:
$k = \frac{61.8}{68.7} \approx 0.89956$

The least squares approximation for k is approximately 0.90.

Quiz 9: Least Squares Approximation

We need to find the linear least squares polynomial approximation, $P_1(x) = a_0 + a_1 x$, for the function $f(x) = x^2 - 2x + 3$ on the interval $[-1, 1]$.

This requires minimizing the error $E = \int_{-1}^{1} [f(x) - P_1(x)]^2 dx$. The coefficients a_0 and a_1 are found by solving the following system of normal equations:

1. $a_0 \int_{-1}^{1} 1\, dx + a_1 \int_{-1}^{1} x\, dx = \int_{-1}^{1} f(x)\, dx$
2. $a_0 \int_{-1}^{1} x\, dx + a_1 \int_{-1}^{1} x^2\, dx = \int_{-1}^{1} x f(x)\, dx$

First, we evaluate the integrals on the left side:

- $\int_{-1}^{1} 1\, dx = [x]_{-1}^{1} = 2$
- $\int_{-1}^{1} x\, dx = [\frac{x^2}{2}]_{-1}^{1} = 0$
- $\int_{-1}^{1} x^2\, dx = [\frac{x^3}{3}]_{-1}^{1} = \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}$

Next, we evaluate the integrals on the right side with $f(x) = x^2 - 2x + 3$:

- $\int_{-1}^{1} (x^2 - 2x + 3)\, dx = [\frac{x^3}{3} - x^2 + 3x]_{-1}^{1} = (\frac{1}{3} - 1 + 3) - (-\frac{1}{3} - 1 - 3) = \frac{7}{3} + \frac{13}{3} = \frac{20}{3}$
- $\int_{-1}^{1} x(x^2 - 2x + 3)\, dx = \int_{-1}^{1} (x^3 - 2x^2 + 3x)\, dx = [\frac{x^4}{4} - \frac{2x^3}{3} + \frac{3x^2}{2}]_{-1}^{1} = (\frac{1}{4} - \frac{2}{3} + \frac{3}{2}) - (\frac{1}{4} + \frac{2}{3} + \frac{3}{2}) = -\frac{4}{3}$

Now, substitute these values back into the normal equations:

1. $a_0(2) + a_1(0) = \frac{20}{3} \implies 2a_0 = \frac{20}{3} \implies a_0 = \frac{10}{3}$
2. $a_0(0) + a_1(\frac{2}{3}) = -\frac{4}{3} \implies \frac{2}{3} a_1 = -\frac{4}{3} \implies a_1 = -2$

The linear least squares polynomial approximation is:
$P_1(x) = \frac{10}{3} - 2x$

Quiz 10: Chebyshev Polynomial

We want to show that for each Chebyshev polynomial $T_n(x)$, the following identity holds:
$\int_{-1}^{1} \frac{(T_n(x))^2}{\sqrt{1-x^2}} dx = \frac{\pi}{2}$
This statement is true for $n \geq 1$. A different result is obtained for $n = 0$.

Proof for $n \geq 1$:

1. Start with the definition of the Chebyshev polynomial, $T_n(x) = \cos(n \arccos x)$.
2. Perform a change of variables in the integral. Let $x = \cos(\theta)$, which implies $\theta = \arccos(x)$.
   - The differential becomes $dx = -\sin(\theta)d\theta$.
   - The limits of integration change from $x = -1$ to $\theta = \pi$ and from $x = 1$ to $\theta = 0$.
   - The term $\sqrt{1 - x^2}$ becomes $\sqrt{1 - \cos^2 \theta} = \sin \theta$ for $\theta \in [0, \pi]$.
3. Substitute these into the integral:
   $\int_{\pi}^{0} \frac{(\cos(n\theta))^2}{\sin \theta} (-\sin \theta \, d\theta)$
4. Simplify the expression. The $\sin \theta$ terms cancel, and the negative sign from the differential reverses the limits of integration:
   $\int_{0}^{\pi} \cos^2(n\theta) \, d\theta$
5. Use the power-reduction identity $\cos^2(\alpha) = \frac{1+\cos(2\alpha)}{2}$:
   $\int_{0}^{\pi} \frac{1+\cos(2n\theta)}{2} \, d\theta$
6. Evaluate the integral:
   $\frac{1}{2} \left[\theta + \frac{\sin(2n\theta)}{2n} \right]_0^{\pi}$
7. Substitute the limits. Since n is an integer, $\sin(2n\pi) = 0$ and $\sin(0) = 0$:
   $\frac{1}{2} \left[(\pi + \frac{0}{2n}) - (0 + \frac{0}{2n}) \right] = \frac{\pi}{2}$
   This proves the identity for $n \geq 1$.

Case for $n = 0$:

For $n = 0$, $T_0(x) = 1$. The integral becomes:
$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = [\arcsin(x)]_{-1}^{1} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$
Therefore, the property as stated in the quiz holds specifically for $n \geq 1$.

Quiz 11: Composite Numerical Integration

We are given five pieces of information to determine the values of $f(x)$ at $x = -1, -0.5, 0, 0.5, 1$.

1. **Midpoint Rule:** The integral $\int_{-1}^{1} f(x)dx$ gives 12.
   - The formula is $(b - a)f(\frac{a+b}{2})$.
   - $(1 - (-1))f(\frac{-1+1}{2}) = 2f(0)$.
   - $2f(0) = 12 \implies \mathbf{f(0) = 6}$.
2. **Composite Midpoint Rule ($n = 2$):** The integral gives 5.
   - The interval $[-1, 1]$ is split into $[-1, 0]$ and $[0, 1]$, with midpoints at -0.5 and 0.5. The step size $h = 1$.
   - The formula is $h(f(-0.5) + f(0.5))$.
   - $1 \cdot (f(-0.5) + f(0.5)) = 5$.
3. **Composite Simpson's Rule ($n = 2$):** The integral gives 6.
   - The interval $[-1, 1]$ uses nodes $x_0 = -1, x_1 = 0, x_2 = 1$. The step size $h = 1$.
   - The formula is $\frac{h}{3}(f(x_0) + 4f(x_1) + f(x_2))$.
   - $\frac{1}{3}(f(-1) + 4f(0) + f(1)) = 6$.

4. **Given Condition 1:** $f(-1) = f(1)$.
5. **Given Condition 2:** $f(-0.5) = f(0.5) - 1$.

Now we solve the system of equations:

- From (1), we have $f(0) = 6$.
- Substitute condition (5) into the equation from (2):
  - $(f(0.5) - 1) + f(0.5) = 5$
  - $2f(0.5) = 6 \implies \mathbf{f(0.5) = 3}$.
- Using this result in condition (5):
  - $f(-0.5) = 3 - 1 \implies \mathbf{f(-0.5) = 2}$.
- Substitute $f(0) = 6$ and $f(-1) = f(1)$ (condition 4) into the equation from (3):
  - $\frac{1}{3}(f(1) + 4(6) + f(1)) = 6$
  - $2f(1) + 24 = 18$
  - $2f(1) = -6 \implies \mathbf{f(1) = -3}$.
- From condition (4), $\mathbf{f(-1) = -3}$.

The determined values are:

- $f(-1) = -3$
- $f(-0.5) = 2$
- $f(0) = 6$
- $f(0.5) = 3$
- $f(1) = -3$

Quiz 12: Gaussian Quadrature

We need to show that a quadrature formula $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$ cannot have a degree of precision greater than $2n - 1$. The degree of precision is the highest degree of a polynomial for which the formula is exact.

Proof by Construction:

1. Let the n distinct points used by the quadrature formula be $x_1, x_2, \ldots, x_n$.
2. Construct a special polynomial $P(x)$ of degree $2n$ using these points:
   $P(x) = (x - x_1)^2 (x - x_2)^2 \cdots (x - x_n)^2 = \left[\prod_{i=1}^{n} (x - x_i)\right]^2$
3. Let's analyze the exact integral of this polynomial over an interval $[a, b]$ with a standard weight function $w(x) > 0$.
   $\int_a^b w(x) P(x)\, dx = \int_a^b w(x) \left[\prod_{i=1}^{n} (x - x_i)\right]^2 dx$
   Since $w(x) > 0$ and the term in the brackets is squared, the integrand is non-negative. Because the integrand is not identically zero, the exact integral is strictly positive.
   $\int_a^b w(x) P(x)\, dx > 0$
4. Now, let's apply the quadrature formula to our polynomial $P(x)$:
   $Q(P) = \sum_{i=1}^{n} c_i P(x_i)$
5. By the way we constructed $P(x)$, its roots are exactly the nodes $x_1, x_2, \ldots, x_n$. Therefore, when we evaluate $P(x)$ at any of these nodes x_i, the result is zero.
   $P(x_i) = 0$ for all $i = 1, \ldots, n$
6. Substituting this into the quadrature formula gives:
   $Q(P) = \sum_{i=1}^{n} c_i (0) = 0$

7. We have found a polynomial $P(x)$ of degree $2n$ for which the exact integral is greater than zero, while the quadrature approximation is exactly zero.
$\int_a^b w(x)P(x)\,dx \neq Q(P)$
Since the formula is not exact for this polynomial of degree $2n$, its degree of precision must be less than $2n$. Thus, the highest possible degree of precision is $2n-1$.

Quiz 13: Runge-Kutta Methods

We must first show that the Midpoint method and the Modified Euler method produce identical results for the initial value problem (IVP) $y' = -y + t + 1$. Then, we explain why this occurs.

Let the IVP be defined by $f(t, y) = -y + t + 1$.

1. Midpoint Method

The formula is $w_{i+1} = w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i))$.

First, evaluate the inner term:
$w_i + \frac{h}{2}f(t_i, w_i) = w_i + \frac{h}{2}(-w_i + t_i + 1)$

Next, substitute this into the outer function evaluation:
$f(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)) = -[w_i + \frac{h}{2}(-w_i + t_i + 1)] + (t_i + \frac{h}{2}) + 1$
$= -w_i + \frac{h}{2}w_i - \frac{h}{2}t_i - \frac{h}{2} + t_i + \frac{h}{2} + 1$
$= -(1 - \frac{h}{2})w_i + (1 - \frac{h}{2})t_i + 1$

Finally, substitute this back into the full method's formula:
$w_{i+1} = w_i + h[-(1 - \frac{h}{2})w_i + (1 - \frac{h}{2})t_i + 1]$
$w_{i+1} = (1 - h + \frac{h^2}{2})w_i + (h - \frac{h^2}{2})t_i + h$

2. Modified Euler Method

The formula is a predictor-corrector sequence:
$w_{i+1}^* = w_i + hf(t_i, w_i)$
$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_{i+1}^*)]$

First, find the predictor w_{i+1}^*:
$w_{i+1}^* = w_i + h(-w_i + t_i + 1) = (1 - h)w_i + ht_i + h$

Next, evaluate f at the predicted point:
$f(t_{i+1}, w_{i+1}^*) = -w_{i+1}^* + t_{i+1} + 1 = -[(1 - h)w_i + ht_i + h] + (t_i + h) + 1$
$= -(1 - h)w_i - ht_i - h + t_i + h + 1$
$= -(1 - h)w_i + (1 - h)t_i + 1$

Finally, use the corrector formula:
$w_{i+1} = w_i + \frac{h}{2}[(-w_i + t_i + 1) + (-(1 - h)w_i + (1 - h)t_i + 1)]$
$w_{i+1} = w_i + \frac{h}{2}[(-1 - 1 + h)w_i + (1 + 1 - h)t_i + 2]$
$w_{i+1} = w_i + (\frac{h^2}{2} - h)w_i + (h - \frac{h^2}{2})t_i + h$
$w_{i+1} = (1 - h + \frac{h^2}{2})w_i + (h - \frac{h^2}{2})t_i + h$

The resulting expressions for w_{i+1} from both methods are identical.

Reason:

This equivalence occurs because the function $f(t, y) = -y + t + 1$ is linear with respect to both y and t. The Midpoint and Modified Euler methods are both second-order Runge-Kutta methods. The terms in the local truncation error that differentiate various second-order methods depend on the second-order partial derivatives of f. For this linear function, all second-order partial derivatives (f_{tt}, f_{ty}, f_{yy}) are zero. As a result, the distinguishing error terms vanish, and all second-order Runge-Kutta methods yield the same result for this specific type of IVP.

Quiz 14: Multistep Methods

To derive the Adams-Bashforth two-step explicit method, we start with the exact solution expressed as an integral:
$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} y'(t)\,dt = y(t_i) + \int_{t_i}^{t_{i+1}} f(t, y(t))\,dt$

The core idea is to replace $f(t, y(t))$ with a polynomial approximation. For the two-step method, we use a first-degree polynomial $P_1(t)$ that interpolates the function values (slopes) at two preceding points, t_i and t_{i-1}. Let $f_k = f(t_k, w_k)$.

Using Newton's backward-difference formula, the interpolating polynomial is:
$P_1(t) = f_i + \frac{f_i - f_{i-1}}{t_i - t_{i-1}}(t - t_i)$

For a constant step size h, $t_i - t_{i-1} = h$. We can make a substitution $t = t_i + sh$, which gives $dt = h\,ds$. The limits of integration change from $t \in [t_i, t_{i+1}]$ to $s \in [0, 1]$.
The integral becomes:
$\int_{t_i}^{t_{i+1}} P_1(t)\,dt = \int_0^1 \left(f_i + \frac{f_i - f_{i-1}}{h}(sh)\right) h\,ds = h\int_0^1 (f_i + s(f_i - f_{i-1}))\,ds = h\left[sf_i + \frac{s^2}{2}(f_i - f_{i-1})\right]_0^1$
$= h\left(f_i + \frac{1}{2}(f_i - f_{i-1})\right) = h\left(\frac{3}{2}f_i - \frac{1}{2}f_{i-1}\right)$

Replacing the integral in the original expression with this approximation gives the Adams-Bashforth two-step method:
$w_{i+1} = w_i + \frac{h}{2}(3f(t_i, w_i) - f(t_{i-1}, w_{i-1}))$

Quiz 15: Stability

To investigate the stability of the Trapezoidal method, we apply it to the standard test equation $y' = \lambda y$, where λ is a complex number with $Re(\lambda) < 0$.

The Trapezoidal method is:
$w_{i+1} = w_i + \frac{h}{2}(f(t_i, w_i) + f(t_{i+1}, w_{i+1}))$

For the test equation, $f(t, y) = \lambda y$. Substituting this into the formula gives:
$w_{i+1} = w_i + \frac{h}{2}(\lambda w_i + \lambda w_{i+1})$

Since this method is implicit, we must solve for w_{i+1}:
$w_{i+1} - \frac{h\lambda}{2}w_{i+1} = w_i + \frac{h\lambda}{2}w_i w_{i+1}\left(1 - \frac{h\lambda}{2}\right) = w_i\left(1 + \frac{h\lambda}{2}\right)w_{i+1} = \frac{1+h\lambda/2}{1-h\lambda/2}w_i$

The method is stable if the magnitude of the amplification factor, $Q(h\lambda) = \frac{1+h\lambda/2}{1-h\lambda/2}$, is less than or equal to 1.
Let $z = h\lambda$. The condition is $|Q(z)| \le 1$.
$\left|\frac{1+z/2}{1-z/2}\right| \le 1 \implies |1 + z/2| \le |1 - z/2|$

Let $z = x + iy$.
$|1 + \frac{x+iy}{2}| \le |1 - \frac{x+iy}{2}||(1 + \frac{x}{2}) + i\frac{y}{2}|^2 \le |(1 - \frac{x}{2}) - i\frac{y}{2}|^2(1 + \frac{x}{2})^2 + (\frac{y}{2})^2 \le (1 - \frac{x}{2})^2 + (\frac{y}{2})^2$
$1 + x + \frac{x^2}{4} \le 1 - x + \frac{x^2}{4}x \le -x \implies 2x \le 0 \implies x \le 0$

The stability condition is $Re(z) \le 0$, which means $Re(h\lambda) \le 0$.

Conclusion on Stability:

The region of absolute stability is the entire left-half of the complex plane, including the imaginary axis. This means that for any stable differential equation (where $Re(\lambda) < 0$) and any positive step size $h > 0$, the numerical method will be stable. A method with this property is called A-stable. The Trapezoidal method is A-stable, which is a highly desirable characteristic for numerical methods used to solve stiff differential equations.